Exercise 18

Prove the statement using the ε, δ definition of a limit and illustrate with a diagram like Figure 9.

$$\lim_{x \to -2} (3x + 5) = -1$$

Solution

According to Definition 2, proving this limit is logically equivalent to proving that

$$\text{if} \quad |x - (-2)| < \delta \qquad \text{then} \qquad |(3x + 5) - (-1)| < \varepsilon$$

for all positive ε. Start by working backwards, looking for a number δ that's greater than $|x + 2|$.

$$|(3x + 5) - (-1)| < \varepsilon$$

$$|3x + 6| < \varepsilon$$

$$|3(x + 2)| < \varepsilon$$

$$3|x + 2| < \varepsilon$$

$$|x + 2| < \frac{\varepsilon}{3}$$

Choose $\delta = \varepsilon/3$. Now, assuming that $|x + 2| < \delta$,

$$\begin{aligned} |(3x + 5) - (-1)| &= |3x + 6| \\ &= |3(x + 2)| \\ &= 3|x + 2| \\ &< 3\delta \\ &= 3\left(\frac{\varepsilon}{3}\right) \\ &= \varepsilon. \end{aligned}$$

Therefore, by the precise definition of a limit,

$$\lim_{x \to -2} (3x + 5) = -1.$$

Below is an illustration like Figure 9.